Linear System Theory - lecture 11

SISO polynomial controllers

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3/12/20

outline SISO polynomial setting.

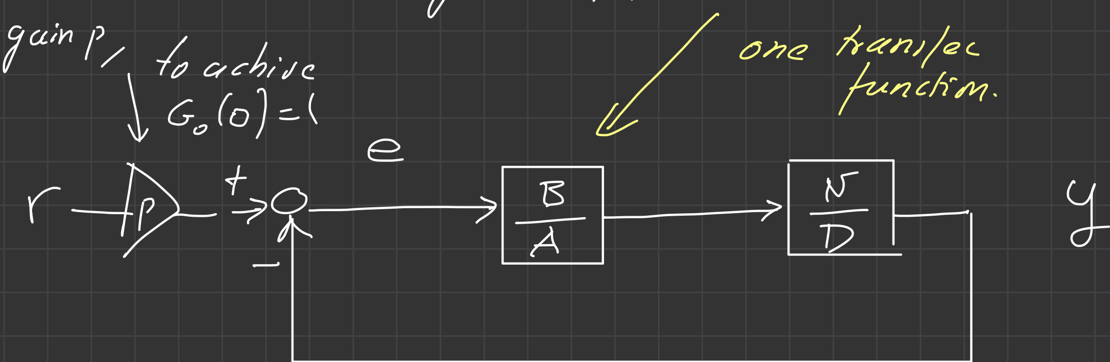
1. pole assignment (Sylvester matrix)

2. disturbance rejection and tracking (internal model principle)

3. model matching (2 d° of-freedom controller)

1. Pole placement

single d° of freedom controller.



$$G_o = \frac{p\,\frac{B}{A}\frac{N}{D}}{1+\frac{B}{A}\frac{N}{D}} = \frac{p\,BN}{AD+BN}$$

- zeros of $N(s)$ are maintained
- new zeros are introduced $B(s)$
- poles are shifted from AD to $AD + BN$

$\frac{N}{D}$ proper and coprime $\quad d^\circ D = n$

$\frac{B}{A}$ proper of $\quad d^\circ A = m$

$$D(s) = D_0 + D_1 s + D_2 s^2 \dots + D_n s^n$$
$$N(s) = N_0 + N_1 s + N_2 s^2 + \dots + N_n s^n$$

$$B(s) = B_0 + B_1 s + \dots + B_m s^m$$
$$A(s) = A_0 + A_1 s + \dots + A_m s^m$$

$$m = ?$$

One needs to assign

$$AD + BN = F(s)$$

$$A_0 D_0 + B_0 N_0$$

$$A_0 D_1 + B_0 N_1 + A_1 D_0 + B_1 N_0$$

$$A_0 D_2 + B_0 N_2 + A_1 D_1 + B_1 N_1 + A_2 D_0 + B_2 N_0$$

$$A_m D_m + B_m N_m$$

$$\boxed{\begin{bmatrix} A_0 & B_0 & A_1 & B_1 & A_2 & B_2 & \cdots & A_m & B_m \end{bmatrix} S_m = \begin{bmatrix} F_0 & F_1 & \cdots & F_{n+m} \end{bmatrix}}$$

Sylvester matrix

$$2(m+1)\left\{ \underbrace{\begin{bmatrix} D_0 & D_1 & D_2 & D_3 & \cdots & D_n & 0 & \cdots & 0 \\ N_0 & N_1 & N_2 & N_3 & \cdots & N_n & & & \\ 0 & D_0 & D_1 & D_2 & \cdots & & D_n & 0 \cdots & 0 \\ 0 & N_0 & N_1 & N_2 & \cdots & & N_n & 0 \cdots & 0 \\ & & \ddots & & & & & & \\ & & & & & & & & \end{bmatrix}}_{n+m+1} \right. := \boxed{S_m}$$

Full column rank $\iff$ a solution for any $F(s)$

$$2(m+1) \geq n+m+1$$

if row rank $S_m < n$, not all $F(s)$ can be realized.

row rank $S_m > n$ all $F(s)$ can be assigned and in many ways.

row rank $S_m = n$ unique solution.

Square matrix

$m = n-1$

Thm: $\frac{N}{D}$ proper & coprime

$\deg N < \deg D = n$

$m \geq n-1$

Then for any polynomial of degree $n+m$, $F(s)$, there exits a proper compensator

$C(s) = B(s)/A(s)$ of degree m

such that the overall transfer function equals

$$G_o(s) = \frac{p\,N(s)B(s)}{F(s)}$$

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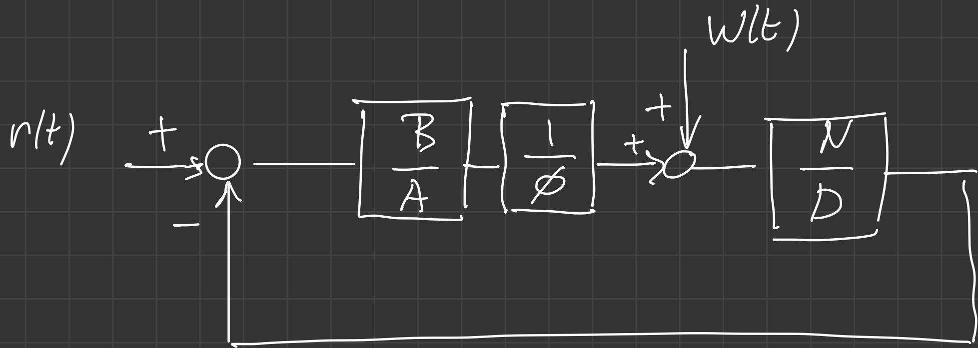
2. Tracking and disturbance rejection



$$W(s) = \frac{N_w}{D_w} \qquad R(s) = \frac{N_r}{D_r} \quad \text{are known.}$$

$$G_{yw} = \frac{N/D}{1 + \frac{B}{A}\frac{1}{\phi}\frac{N}{D}} = \frac{NA\phi}{A\phi D + BN}$$

$$G_{yr} = \frac{\frac{B}{A}\frac{1}{\phi}\frac{N}{D}}{1 + \frac{B}{A}\frac{1}{\phi}\frac{N}{D}} = \frac{BN}{AD\phi + BN}$$

Response to $w(t)$

$$Y_w(t) = \frac{NA\phi}{AD\phi + BN} \frac{N_w}{D_w}$$

Response to $r(t)$

$$Y_r(t) = \frac{BN}{AD\phi + BN} \frac{N_r}{D_r}$$

tracking error:

$$e = (1 - G_{yr}) r = \frac{AD\phi}{AD\phi + BN} \frac{N_r}{D_r}$$

Thm: Let $\phi(s)$ be the least common multiple of the unstable poles of $R(s)$ $W(s)$. If no root of $\phi(s)$ is a zero of $G(s) = \frac{N(s)}{D(s)}$, then there exists a proper compensator such that the overall system will track $r(t)$ and reject $w(t)$.

$$A(s) \underbrace{D(s)\phi(s)}_{\tilde{D}(s)} + B(s) N(s) = F(s)$$

Proceed as before using $\tilde{D}(s)$ instead of $D(s)$.

$D(s)\,\phi(s)$ and $N(s)$ are coprime
(no root is a zero)

- all unstable roots of $D_W(s)$ and $D_r(s)$ are cancelled.

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3. Model matching (2 d° of-freedom controller, changing the zeros and the poles)

1. proper rational $G_o(s)$
2. All forward paths from r to y must go through the plant $G(s) = \frac{N(s)}{D(s)}$
3. BIBO stability of every possible input-output pair

Thm: ** $G(s) = \frac{N(s)}{D(s)}$ available transfer function

$G_o(s) = \frac{E(s)}{F(s)}$ desired transfer function.

1. All roots of $F(s)$ have negative real parts.

2. Pole-zero excess inequality:

$$\deg F(s) - \deg E(s) \geq \deg D(s) - \deg N(s)$$

3. All zeros of $N(s)$ with zero or positive real parts are retained in $E(s)$.

Thm (*) $\quad T(s) \triangleq \dfrac{G_o(s)}{G(s)} = \dfrac{E(s)\,D(s)}{F(s)\,N(s)}$

is proper and BIBO stable.

(2') Properness gives 2. $\quad \deg F + \deg N \geq \deg E + \deg D$

then zeros of $N(s)$ must be cancelled by zeros of $E(s)$. Thus thm (**) follows from thm (*).

(1)

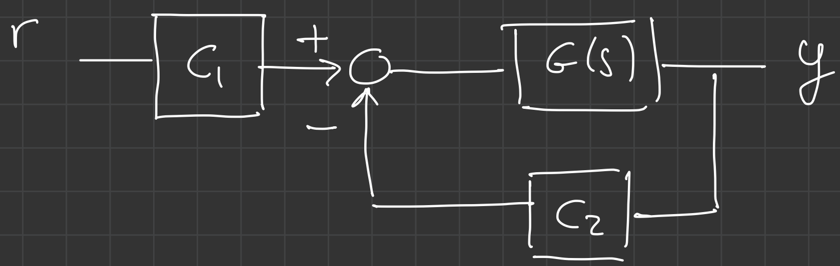
$$\left| \begin{array}{l} Y(s) = G_o\, R(s) = G(s)\, U(s) \\ U(s) = \dfrac{G_o}{G} R(s) = T(s)\, R(s) \end{array} \right.$$

$T(s)$ must be proper

Thus, $G_o(s)$ and $T(s)$ must be BIBO stable.

(2) this proves the necessity of the theorem.

2- d° of freedom controller:



$$C_1(s) = \frac{L(s)}{A_1(s)} \qquad C_2(s) = \frac{M(s)}{A_2(s)}$$

We assume $A_1(s) = A_2(s) = A(s)$ because model matching can still be achieved.

$$G_o = C_1 \frac{G}{1+C_2 G} = \frac{LN}{AD + MN}$$

$$G_o = \frac{E(s)}{F(s)} = \frac{L\,N}{AD + MN}$$

1. compute $\dfrac{G_0}{N(s)} = \dfrac{E}{FN} = \dfrac{\bar{E}}{\bar{F}}$

(cancel common factors between N and E)

$$G_0 = \frac{\bar{E}N}{\bar{F}} = \frac{LN}{AD+MN}$$

tempted to set $\bar{E} = L$, but this might not yield a proper C_2.

2. Introduce an arbitrary Hurwitz polynomial $\hat{F}(s)$, such that

$$\boxed{\text{deg. } \bar{F}\hat{F} \text{ is } 2n-1 \text{ or higher}}$$

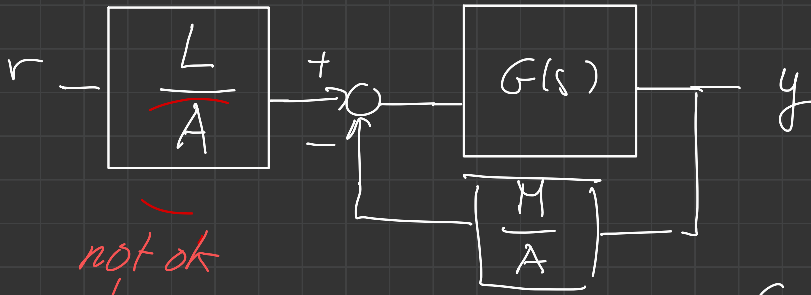
$$G_0 = \frac{\bar{E}\hat{F}N}{\bar{F}\hat{F}} = \frac{LN}{AD+MN}$$

Set $\boxed{L = \bar{E}\hat{F}}$

and solve

$$AD + MN = \bar{F}\hat{F}$$

as usual using the sylvester matrix.

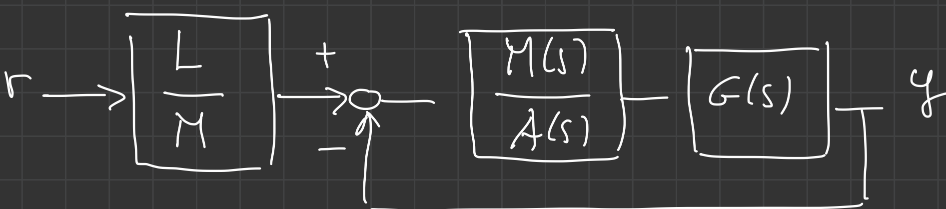


not ok
because no
loop to stabilize $\frac{1}{A}$

$$G_0 = \frac{\frac{N}{D}}{1+\frac{NM}{DA}} \cdot \frac{L}{A}$$

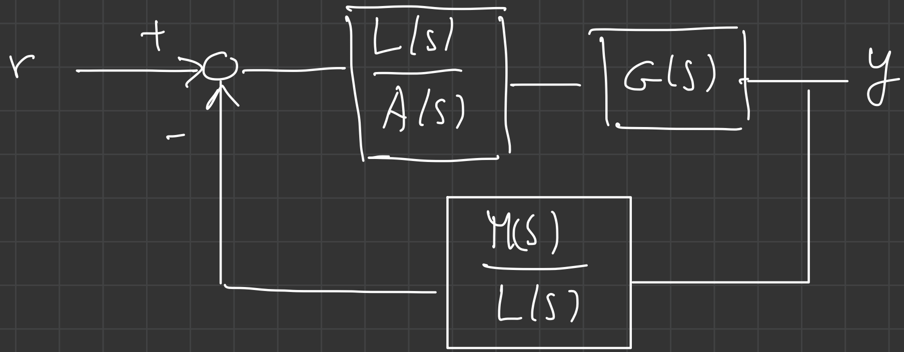
$$= \frac{L}{A} \cdot \frac{NA}{AD+MA}$$

$A(s)$ may not be Hurwitz.



no loop around $M(s)$
no direct control over $M(s)$; could also be unstable.
However $A(s)$ is stabilized (in case it is unstable) by the plant.

$$G_0 = \frac{L}{M} \frac{\frac{MN}{AD}}{1+\frac{MN}{AD}} = \frac{L}{M} \frac{MN}{AD+MN}$$



$$G_0 = \frac{\frac{L}{A}\frac{N}{D}}{1 + \frac{LN}{AD}\frac{M}{L}} = \frac{LN}{AD + MN}$$

pole zero cancellation occurs in $\frac{L}{A}$

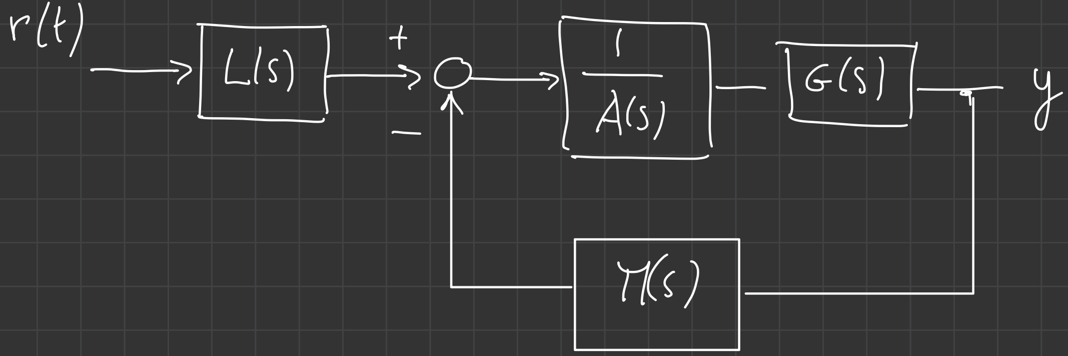
since $L = \hat{F}\bar{E}$.

However $\frac{M(s)}{L(s)}$ may not be proper

Another drawback

$\frac{L(s)}{A(s)}$ & $\frac{M(s)}{L(s)}$ have different denominators

their realizations requires more states.
(2m integrators).



However, this means using polynomials in $M(s)$! which can amplify noise (differentiators derivatives)

$L(s)$ is not a problem but we need to know it ahead of time (know $r(t)$ and their derivatives at time t).

$$G_0 = L \cdot \frac{\frac{1}{A}\frac{N}{D}}{1 + M\left(\frac{1}{A}\frac{N}{D}\right)} = L \cdot \frac{N}{AD + MN}$$